

A CHERN CHARACTER IN CYCLIC HOMOLOGY

LUCA QUARDO ZAMBONI

ABSTRACT. We show that inner derivations act trivially on the cyclic cohomology of the normalized cyclic complex $\mathcal{E}(\Omega)/\mathcal{D}(\Omega)$ where Ω is a differential graded algebra. This is then used to establish the fact that the map introduced in [GJ] defines a Chern character in K theory.

J.-M. Bismut recently extended the classical theory of Chern characters to the equivariant theory of the free loop space LX ; he explicitly constructed an S^1 -invariant equivariantly closed form $\omega(E, \nabla)$ on LX whose restriction to X , the space of constant loops, is $\text{Ch}(E) = \text{tr}(\exp \nabla^2)$. Closely related to the equivariant theory of the loop space of X is A. Connes' cyclic theory of the differential graded algebra Ω of differential forms on X . More precisely, Chen's theory of iterated integrals allows us to view cyclic cycles of forms on X as equivariantly closed forms on the loop space LX . This is a generalization of the point of view that a one-form ω on X can be viewed as a function on LX by $\gamma \rightarrow \int_\gamma \omega$.

This suggests the possibility of interpreting the classical theory of Chern characters in the context of cyclic homology i.e., of extending $\text{Ch}(E)$ off the fixed point set X of the circle action on LX by way of a cyclic cycle over Ω . In [GJP], Getzler, Jones, and Petrack, gave an explicit formula for a cyclic cycle $\psi(E, e)$ which via iterated integrals maps to the Bismut Chern character $\Omega(E)$ over LX . A priori, their construction of $\psi(E, e)$ appears to depend on the choice of idempotent e describing the vector bundle E , but in fact, it is shown that $\psi(E, e)$ depends only on the isomorphism class of E in $K_0(X)$, and hence defines the desired Chern character. This is essentially proved by establishing the triviality of certain group actions on the cyclic homology of a differential graded algebra Ω . These actions are natural generalizations of the usual adjoint action $\text{Ad}(g)$, given by conjugation by an invertible element g , and its infinitesimal analogue ad , given by inner derivation, to Connes' cyclic bar complex of Ω .

1. CYCLIC HOMOLOGY

In this section we review the definition of the cyclic homology of a DGA.

According to C. Kassel, by a *mixed complex* we mean a triplet $(\mathcal{E}^*; d; B)$ where $\mathcal{E}^*$ is a graded algebra and where $d: \mathcal{E}^* \rightarrow \mathcal{E}^{*+1}$, and $B: \mathcal{E}^* \rightarrow \mathcal{E}^{*-1}$, are derivations of degrees $+1$ and -1 respectively, and subject to the rela-

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tion $[d, B] = 0$. The associated chain complex $(\mathcal{E}_*^\lambda; d + B)$, called the *cyclic complex* is defined as follows:

$$\mathcal{E}_n^\lambda = \mathcal{E}^n \oplus \mathcal{E}^{n+2} \oplus \mathcal{E}^{n+4} \oplus \dots,$$

and

$$d_n : \mathcal{E}_n^\lambda \rightarrow \mathcal{E}_{n-1}^\lambda,$$

by

$$d_n(\omega^n, \omega^{n+2}, \omega^{n+4}, \dots) = (B\omega^n, d\omega^n + B\omega^{n+2}, d\omega^{n+2} + B\omega^{n+4}, \dots),$$

in short $d_n = d + B$. We define the *cyclic homology* $HC_*(\mathcal{E}^*)$ of the mixed complex $(\mathcal{E}^*; d; B)$ to be the homology of the associated chain complex. Similarly, we define the *even* and *odd cyclic homology groups* $HC^{\text{even}}(\mathcal{E}^*)$ and $HC^{\text{odd}}(\mathcal{E}^*)$.

We now apply this construction to define the *cyclic homology theories* of a DGA. Let $(\Omega; d)$ be a DGA over $\mathbb{C}$ with unit 1. Then we set

$$\mathcal{E}(\Omega) = \sum_p \Omega \otimes (I\Omega)^{\otimes p},$$

where $I\Omega = \Omega/\mathbb{C}$. We make $\mathcal{E}(\Omega)$ into a graded algebra by defining

$$\deg(\omega_0 \otimes \dots \otimes \omega_k) = \sum_{i=0}^k |\omega_i| - k.$$

(The product structure is of course given by the shuffle product.) Next, we define operators d and b

$$d, b : \mathcal{E}^p(\Omega) \rightarrow \mathcal{E}^{p+1}(\Omega),$$

by

$$d(\omega_0 \otimes \dots \otimes \omega_k) = - \sum_{i=0}^k (-1)^{\varepsilon_{i-1}} \omega_0 \otimes \dots \otimes \omega_{i-1} \otimes d\omega_i \otimes \omega_{i+1} \otimes \dots \otimes \omega_k,$$

and

$$\begin{aligned} b(\omega_0 \otimes \dots \otimes \omega_k) = & - \sum_{i=0}^k (-1)^{\varepsilon_i} \omega_0 \otimes \dots \otimes \omega_i \omega_{i+1} \otimes \dots \otimes \omega_k \\ & + (-1)^{(|\omega_k| - 1)\varepsilon_{k-1}} \omega_k \omega_0 \otimes \omega_1 \otimes \dots \otimes \omega_{k-1}, \end{aligned}$$

where

$$\varepsilon_i = |\omega_0| + \dots + |\omega_i| - i.$$

Then, it is easy to show that $(d + b)^2 = 0$, in fact that $d^2 = b^2 = db + bd = 0$. We also define the *A. Connes operator* $B_0 : \mathcal{E}^p(\Omega) \rightarrow \mathcal{E}^{p-1}(\Omega)$ by the formula

$$B(\omega_0 \otimes \dots \otimes \omega_k) = \sum_{i=0}^k (-1)^{(\varepsilon_{i-1} + 1)(\varepsilon_k - \varepsilon_{i-1})} 1 \otimes \omega_i \otimes \dots \otimes \omega_k \otimes \omega_0 \otimes \dots \otimes \omega_{i-1}.$$

It is easily seen that $[B; d + b] = 0$ so that $(\mathcal{E}(\Omega); d + b; B)$ constitutes a mixed complex. Thus we define the *cyclic homology theory* of Ω as $HC(\Omega) = HC(\mathcal{E}(\Omega))$.

For a DGA Ω we define $\mathcal{D}(\Omega)$ to be the subspace of $\mathcal{E}(\Omega)$ generated by the images of the operators $S_i(f)$ and $R_i(f)$, where f is of degree 0 and

$$S_i(f)(\omega_0 \otimes \cdots \otimes \omega_k) = \omega_0 \otimes \cdots \otimes \omega_i \otimes f \otimes \omega_{i+1} \otimes \cdots \otimes \omega_k,$$

and

$$R_i(f) = [(b+d), S_i(f)].$$

Then, it is easy to show that the differentials $(b+d)$ and B map $\mathcal{D}(\Omega)$ into itself and hence $(\mathcal{D}(\Omega); (b+d); B)$ is a sub mixed complex of $(\mathcal{E}(\Omega); (b+d); B)$. Chen's normalized cyclic complex is then defined to be the quotient complex $\mathcal{E}(\Omega)/\mathcal{D}(\Omega)$.

2. THE CONSTRUCTION OF A CYCLIC CYCLE

Let X be a smooth manifold, E a complex vector bundle over X , and let T denote the unit circle S^1 . In this section, we construct an element $\psi(E, e)$ in the complex $\mathcal{E}(\Omega)$ where

$$\Omega = \Omega(X) \otimes \Omega_T(T) = \Omega_T(X \times T),$$

where T acts trivially on X and by multiplication on T . We shall show that $\psi(E, e)$ is $(d+b)+B$ closed in the reduced complex $\overline{\mathcal{E}(\Omega)} = \mathcal{E}(\Omega)/\mathcal{D}(\Omega)$ and hence defines a class in HC^{even} of the normalized complex. (Unfortunately, $\psi(E, e)$ is not closed in $\mathcal{E}(\Omega)$.) Moreover, $\psi(E, e)$ is an extension of the classical Chern character $\text{Ch}(E)$; that is, if we view $\psi(E, e)$ as an equivariantly closed form on the free loop space LX (via the mapping σ) and restrict it to X , the space of constant loops, we obtain $\text{Ch}(E)$. The above construction is due to Getzler, Jones, and Petrack [GJP].

A general element of Ω is of the form $\omega = \alpha + \beta dt$, where $\alpha, \beta \in \Omega(X)$ and where dt is the standard 1-form on the unit circle T . Then we define the differential d_T on Ω by the following rule:

$$d_T(\alpha + \beta dt) = d\alpha + (-1)^{\deg \beta} \beta + d\beta dt.$$

Let E be a vector bundle over X given by an idempotent e , and let ∇_e denote the Levi-Civita connection on E . So, $\nabla_E = ed$. Let $E^\perp$ denote the vector bundle determined by the idempotent $e^\perp = 1 - e$. Similarly, let $\nabla_{E^\perp}$ denote the Levi-Civita connection on $E^\perp$. Let $\mathcal{A}$ be the matrix of 1-forms on X determined by the connection $\nabla_E \oplus \nabla_{E^\perp}$ on the trivial bundle $X \times \mathbb{C}^n = E \oplus E^\perp$, that is,

$$\mathcal{A} = ed + e^\perp d - d.$$

So if $s \in C^\infty(X)^n$, then we find

$$\begin{aligned} \mathcal{A}s &= ed(es) + e^\perp d(e^\perp s) - ds \\ &= edes + eds + e^\perp de^\perp s + e^\perp ds - ds \\ &= (ede + e^\perp de^\perp)s + eds + (1 - e)ds - ds \\ &= (ede + e^\perp de^\perp)s. \end{aligned}$$

So we find

$$\mathcal{A} = ede + e^\perp de^\perp.$$

Let $\mathcal{R}$ denote the curvature associated to the connection $\nabla_E \oplus \nabla_{E^\perp} = d + \mathcal{A}$ on the trivial bundle $X \times \mathbb{C}^n$. So

$$\mathcal{R} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = dede.$$

Next we set $\mathcal{N} = \mathcal{A} - \mathcal{R} dt$.

Lemma 2.1. (1) $d_T e = [e, \mathcal{N}]$, (2) $d_T \mathcal{N} = -\mathcal{N} \wedge \mathcal{N}$.

Next we define $\mathcal{N}_k \in \mathcal{E}(\Omega)$ by

$$\mathcal{N}_k = e \otimes \underbrace{\mathcal{N} \otimes \mathcal{N} \otimes \cdots \otimes \mathcal{N}}_{k\text{-times}}.$$

Lemma 2.2. $d_T \mathcal{N}_{k-1} = -b\mathcal{N}_k$.

Finally, we define

$$\psi(E, e) = \sum_{k=0}^{\infty} \text{Tr}(\mathcal{N}_k),$$

where Tr denotes the generalized trace map. It follows from the previous Lemma that $(d + b)\psi(E, e) = 0$. However, we note that $B\psi(E, e)$ is not equal to 0 in $\mathcal{E}(\Omega)$. On the other hand, since $B\psi(E, e) \in \mathcal{D}(\Omega)$ it follows that $\psi(E, e)$ is a closed form in the normalized complex $\mathcal{E}(\Omega)/\mathcal{D}(\Omega)$. In what follows, we shall prove that the class of $\psi(E, e)$ is independent of the choice of idempotent e and hence defines a Chern character on the Grothendieck group of vector bundles.

3. TRIVIALITY OF ad ON THE CYCLIC HOMOLOGY OF A DGA

In this section, we extend the definition of the adjoint action to a DGA Ω and prove that the induced action on the cyclic homology of Ω is trivial.

Define for all $X \in \Omega^0$,

$$\begin{aligned} \text{ad}(X)(\omega_0 \otimes \cdots \otimes \omega_k) &= \sum_{i=0}^k \omega_0 \otimes \cdots \otimes [\omega_i, X] \otimes \cdots \otimes \omega_k \\ &\quad + \sum_{i=0}^k \omega \otimes \cdots \otimes \omega_i \otimes dX \otimes \omega_{i+1} \otimes \cdots \otimes \omega_k. \end{aligned}$$

We shall write

$$\text{ad}(X)(\omega_0 \otimes \cdots \otimes \omega_k) = \alpha(X)(\omega_0 \otimes \cdots \otimes \omega_k) + \beta(X)(\omega_0 \otimes \cdots \otimes \omega_k).$$

Then clearly $\text{ad}(X)$ is degree preserving, i.e., $\text{ad}(X) : \mathcal{E}^p(\Omega) \rightarrow \mathcal{E}^p(\Omega)$.

Proposition 3.1. $\text{ad}(X)$ commutes with both the operators B and $d + b$, and hence defines an action of Ω^0 on the cyclic homology of Ω .

Proposition 3.2. $\text{ad}(X)$ acts on the normalized cyclic complex $\mathcal{E}(\Omega)/\mathcal{D}(\Omega)$. In fact it maps $\mathcal{D}(\Omega)$ into itself.

Proof. The proof is clear from the definition of $\text{ad}(X)$, and the fact that $\mathcal{D}(\Omega)$ is generated by the images of the operators $S_i(f)$ and $R_i(f)$ defined in §1.

Proposition 3.3. Define h_X by the following formula:

$$h_X(\omega_0 \otimes \cdots \otimes \omega_k) = - \sum_{i=0}^k (-1)^{e_i} \omega_0 \otimes \cdots \otimes \omega_i \otimes X \otimes \cdots \otimes \omega_k.$$

Then

$$(d + b)h_X + h_X(d + b) = \text{ad}(X),$$

and

$$Bh_X + h_X B = 0.$$

Hence $\text{ad}(X)$ acts trivially on the cyclic homology of the DGA Ω .

4. TRIVIALITY OF Ad ON THE CYCLIC HOMOLOGY OF A DGA

For each $g \in \Omega^{0*}$ we define the mapping $\text{Ad}(g) : \mathcal{C}^p(\Omega) \rightarrow \mathcal{C}^p(\Omega)$, by the formula

$$\begin{aligned} \text{Ad}(g)(\omega_0 \otimes \cdots \otimes \omega_k) = \sum g^{-1} \omega_0 \underbrace{g^{-1} dg \otimes \cdots \otimes g^{-1} dg}_{i_0} \otimes \cdots \\ \otimes g^{-1} \omega_k g \otimes \underbrace{g^{-1} dg \otimes \cdots \otimes g^{-1} dg}_{i_k}, \end{aligned}$$

where the sum runs over all $(i_0, i_1, \dots, i_k) \in \mathbb{N}^k$. Then clearly $\text{Ad}(g)$ is degree preserving.

Proposition 4.1. $\text{Ad}(g)$ commutes both with B and $d + b$ and hence defines an action of Ω^{0*} on the cyclic homology $HC(\Omega)$ of the DGA Ω .

Proof. Clearly $\text{Ad}(g)$ commutes with B . That $\text{Ad}(g)$ also commutes with $d + b$ follows from a straightforward calculation using the following two relations:

- (1) $d(g^{-1}dg) = -g^{-1}dg g^{-1}dg$,
- (2) $d(g^{-1}\omega_i g) = -g^{-1}dg g^{-1}\omega_i g + g^{-1}d\omega_i g + (-1)^{|\omega_i|}g^{-1}\omega_i g g^{-1}dg$.

Proposition 4.2. $\text{Ad}(g)$ acts on the normalized cyclic complex $\mathcal{C}(\Omega)/\mathcal{D}(\Omega)$; in fact, it maps $\mathcal{D}(\Omega)$ into itself.

Proof. Clearly $\text{Ad}(g)S_i(f)$ is contained in $\bigcup_k S_k(g^{-1}fg)$.

Theorem 4.3. $\text{Ad}(g)$ acts trivially on the cyclic homology $HC(\Omega)$ of the DGA Ω .

Proof. Let τ be a $(b + d) + B$ cycle. Then τ will consist of a sum of terms of the form $\omega_0 \otimes \cdots \otimes \omega_k$. We shall begin by replacing $\omega_0 \otimes \cdots \otimes \omega_k$ by

$$\begin{pmatrix} \omega_0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \omega_k & 0 \\ 0 & 0 \end{pmatrix},$$

and g by $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ and show that $\text{Ad}(\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix})$ acts trivially on the cyclic cohomology of the DGA M

$$M = M_{2 \times 2}(\Omega^0) \oplus M_{2 \times 2}(\Omega^1) \oplus M_{2 \times 2}(\Omega^2) \oplus \cdots.$$

This will be enough since the generalized trace map $\text{Tr} : M^{\otimes k} \rightarrow \Omega^{\otimes k}$ defined by

$$\text{Tr}(M^1 \otimes \cdots \otimes M^k) = \sum M_{i_1, i_2}^1 \otimes M_{i_2, i_3}^2 \otimes \cdots \otimes M_{i_k, i_1}^k,$$

induces a homomorphism in homology $HC(M; W) \rightarrow HC(\Omega; W)$.

Let $\mathcal{B}$ denote the algebra $\mathbb{Q}[x, y]/(x^2 + y^2 - 1)$. Then we define

$$m(x, y) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

and observe that $m(0, 1) = I_{2 \times 2}$ while $m(1, 0) = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$.

Let τ now denote a $(b+d)+B$ cycle in $\mathcal{C}(M)$. Then $\omega = \text{Ad}(m(x, y))\tau - \tau$ is a $(b+d)+B$ cycle over the DGA $M \otimes_{\mathbb{Q}} \mathcal{B}$. Then, if we define the derivation D of $\mathcal{B}$ by $D = yd/dx$; then $D(x) = y$ and $D(y) = -x$ and moreover the following relations are easily verified:

$$\begin{aligned} D \begin{pmatrix} x & -y \\ y & x \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \\ D \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} &= - \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ D \begin{pmatrix} x & y \\ -y & x \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \\ D \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} &= - \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

therefore, it follows that

$$\begin{aligned} D\omega &= \text{Ad} \left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) \text{ad} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \text{Ad} \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right) \tau \\ &\quad + \text{Ad} \left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \right) \\ &\quad \times \text{ad} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \text{Ad} \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right) \tau. \end{aligned}$$

Therefore, since ad acts trivially on $HC(M \otimes \mathcal{B})$, it follows that $[D\omega] = 0$ in $HC(M \otimes \mathcal{B})$, where $[D\omega]$ denotes the equivalence class of $D\omega$ in $HC(M \otimes \mathcal{B})$.

But, in view of the identification

$$HC(M \otimes \mathcal{B}) \xrightarrow{\sim} HC(M) \otimes \mathcal{B},$$

where we view $M \otimes \mathcal{B}$ as a DGA over $\mathcal{B}$, we can write

$$[\omega] = \sum_i \alpha_i \otimes \beta_i,$$

where $\alpha_i \in HC(M)$ are linearly independent and $\beta_i \in \mathcal{B}$. Then,

$$[D\omega] = \sum_i \alpha_i \otimes D\beta_i = 0,$$

and so we can conclude that $D\beta_i = 0$ for all i . A straightforward calculation shows that if $\beta \in \mathcal{B} = \mathbb{Q}[x, y]/(x^2 + y^2 - 1)$ and $D\beta = 0$, then β is a constant. This means that the evaluation map is constant on $[\omega] = [\text{Ad}(m(x, y))\tau] - [\tau]$. Since, we say that $m(0, 1) = I_{2 \times 2}$, it follows that $\text{Ad}(m(1, 0)) = \text{Ad}(\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix})$ acts trivially on $HC(M)$.

Theorem 4.4. *The $(d+b)+uB$ cycle $\psi(E, e)$ of the reduced cyclic complex $\mathcal{C}(\Omega)/\mathcal{D}(\Omega)$ defines a Chern character from the Grothendieck group $K_0(X)$, of all isomorphism classes of vector bundles over X , into $HC^{\text{even}}(\Omega)$ where $\Omega = \Omega_i(X \otimes T)$; that is, $\psi(e, E)$ depends only on the conjugacy class of the idempotent e .*

Proof. We recall, that

$$\psi(E, e) = \sum_{k=0}^{\infty} \text{Tr } \mathcal{N}_k,$$

where

$$\mathcal{N}_k = e \otimes \underbrace{\mathcal{N} \otimes \mathcal{N} \otimes \cdots \otimes \mathcal{N}}_k,$$

and where $\mathcal{N} = \mathcal{A} - \mathcal{R} dt$. $\mathcal{A} = ede + e^\perp de^\perp$ and $\mathcal{R} = dede$, are the connection and curvature form respectively of the connection $\nabla_E \oplus \nabla_{E^\perp}$ on the trivial bundle $X \times \mathbb{C}^n$. The transformation $e \rightarrow g^{-1}eg$ leads to the transformations $\mathcal{A} \rightarrow g^{-1}\mathcal{A}g + g^{-1}dg$ and $\mathcal{R} \rightarrow g^{-1}\mathcal{R}g$, and hence $\psi(E, g^{-1}eg) = \text{Ad}(g)\psi(E, e)$. Since $\text{Ad}(g)$ acts trivially on $HC(\Omega)$, it follows that the class of $\psi(E, e)$ is independent of the choice of idempotent e describing the vector bundle E . Moreover, it is easy to see from the definition of $\psi(E, e)$, that

$$\psi\left(E, \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}\right) = \psi(E, e),$$

and that

$$\psi\left(E \oplus F, \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right) = \psi(E, e) + \psi(F, f).$$

Therefore, since $K_0(X)$ is defined as the $\bigcup_n$ idempotents in $M_n(C^\infty(X))$, it follows that $\psi(E, e)$ defines a Chern character on $K_0(X)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TEXAS 76203-5116